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EXISTENCE OF MILD SOLUTIONS IN THE α-NORM FOR SOME PARTIAL FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

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Abstract. In this work, we discuss the existence of mild solutions in the α-norm for some partial functional integrodifferential equations with infinite delay. We assume that the linear part generates an analytic semigroup on a Banach space X and the nonlinear part is a Lipschitz continuous function with respect to the fractional power norm of the linear part.

1. Introduction

Byszewski [11] studied the problem of existence of solution of semilinear evolution equation with nonlocal conditions in Banach spaces. Byszewski and Acka [13] established the existence and uniqueness and continuous dependence of a mild solution of a semilinear functional differential equation with nonlocal condition of the form

$$u'(t) + Au(t) = f(t, u_t), t \in [0, a], \tag{1.1}$$

$$u(s) + [g(u_{t_1}, ..., u_{t_p})](s) = \phi(s), s \in [-r, 0], \tag{1.2}$$

where $0 < t_1 < ... < t_p \leq a (p \in \mathbf{N})$, $-A$ is the infinitestimal generator of a C_0 semigroup of operators on a Banach space, f, g and ϕ are given functions and $u_t(s) = u(t + s)$ for $t \in [0, a], s \in [-r, 0]$.

In this paper, we shall prove the existence and uniqueness of mild solutions in the α-norm of a functional integrodifferential equation with nonlocal conditions of the form

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$$(1.3) \qquad u'(t) + Au(t) = f(t, u_t, \int_0^t k(t, \tau, u_\tau)d\tau), t \in [0, a],$$

$$(1.4) \qquad u(s) + [g(u_{t_1}, ..., u_{t_p})](s) = \phi(s) \in B_\alpha, s \in (-\infty, 0],$$

where $-A$ is the infinitestimal generator of C_0 semigroup of operators $(T(t))_{t\geq 0}$ on a Banach space X and $\phi \in C((-\infty, 0] : X)$ and the nonlinear operators f, k, g are given functions satisfying some assumptions.

Theorems about the existence, uniqueness and stability of solutions of differential, integrodifferential equations and functional-differential abstract evolution equations with nonlocal conditions were studied by Byszewski [11,12,13], Balachandran[5], Chandrasekara [6], and Lin and Lu [18].

2. Preliminaries

Here we assume that X is a Banach space with norm $||\cdot||, -A$ is the infinitestinmal generator of a C_0 semigroup $(T(t))_{t\geq 0}$ on X and

$M = \sup_{t\in[0,a]} ||T(t)||_{B(X)}$. In the sequel the operator norm $||\cdot||_{B(X)}$ will be denoted by $||\cdot||$. To simplify the notation let us take $I_0 = (-\infty, 0], I = [0, a]$ and $E = C((-\infty, 0] : X), Y = C((-\infty, a] : X), Z = C([0, a] : X)$. For a continuous function $w : (-\infty, a] \to X$, we denote w_t a function belong to E and defined by $w_t = w(t + s)$ for $t \in I, s \in I_0$. Let $f : I \times E \times E \to X, k : I \times I \times E \to E$ and $\phi \in E$.

We make the following assumptions:

(A_1) For every $u_t, w_t \in E$ and $t \in I, f(., u_t, w_t) \in X$

(A_2) There exists a constant $L > 0$ such that

$$||f(t, x_t, w_t) - f(t, y_t, u_t)|| \leq L[||x - y||_Y + ||w - u||_Y]$$

for $x, y, w, u \in Y, t \in I$.

(A_3) There exists a constant $K > 0$ such that

$$||k(t, s, x_s) - k(t, s, y_s)|| \leq K||x - y||_Y$$

for $x, y \in Y, s \in I$.

(A_4) Let $g : E^P \to E$ and there exists a constant $G > 0$ such that

$$||[g(w_{t_1}, ..., w_{t_p})](s) - [g(u_{t_1}, ..., u_{t_p})](s)|| \leq G||w - u||_Y$$

for $w, u \in Y, s \in I_0$.

(A_5) $M_\alpha L_a[1 + (1 + aK) \int_0^a \frac{e^{\omega s}}{s^\alpha} ds]||w - u||_\alpha < 1$, where M_α, ω, and L_a are constants to be specified later.

DEFINITION 2.1. ([19]) A function $u \in Y$ satisfying the conditions:

$$(i) \quad u(t) = T(t)\phi(0) - T(t)[g(u_{t_1}, ..., u_{t_p})](0) + \int_0^t T(t-s)f(s, u_s, \int_0^s k(s, \theta, u_\theta)d\theta)ds, t \in I,$$

$$(ii) \quad u(s) + [g(u_{t_1}, ..., u_{t_p})](s) = \phi(s), s \in I_0$$

is said to be *a mild solution* of the nonlocal Cauchy problem.

We will discuss the following abstract partial differential equations with infinite delay:

$$u'(t) + Ax(t) = F(t, u_t), t \in [0, a] \tag{2.1}$$

$$u(s) + [g(u_{t_1}, ..., u_{t_p})](s) = \phi(s), s \in (-\infty, 0], \tag{2.2}$$

where $-A$ generates an analytic semigroup $(T(t))_{t\geq 0}$ on a Banach space X, B is a Banach space of functions mapping $(-\infty, 0]$ to X and satisfying some axioms that will be introduced later. For $0 < \alpha < 1, A^\alpha$ denotes the fractional power of A ; we assume that F is defined on a subspace B_α with values in X, where B_α is defined by

$$B_\alpha = \{\phi \in B : \phi(\theta) \in D(A^\alpha) \text{ for } \theta \leq 0 \text{ and } A^\alpha\phi \in B\},$$

the function $A^\alpha\phi$ is defined by

$$(A^\alpha\phi)(\theta) = A^\alpha(\phi(\theta)) \text{ for } \theta \leq 0.$$

We suppose that F is Lipschitz continuous with respect to the fractional power norm of A^α. For every $t \geq 0$, the history function $x_t \in B_\alpha$ is defined by

$$x_t(\theta) = x(t + \theta) \text{ for } \theta \leq 0.$$

We will discuss the existence of a mild solution in the α-norm for equations(2.1),(2.2). Recall that when f is Lipschitz continuous in B with respect to the X-norm, the equation has been exensively studied by several authors;for more details we refer to [1,5,9] and the references therein.

This work is motivated by the papers of Benkhalti[7] and Balhachandran[4], where the authors studied the existence and stability in the α-norm for partial functional differential equations with finite delay; they assumed that $F : C_\alpha = C([-r, 0] : D(A^\alpha)) \to X$ is continuous, where C_α is the Banach space of continuous functions from $[-r, 0]$ to $D(A^\alpha)$, endowed with the following norm

$$||\phi||_\alpha = \sup_{-r\leq\theta\leq 0} |A^\alpha\phi(\theta)|.$$

The authors investigated several results regarding the existence, the regularity, and the stability of solutions in C_α. Recently, in [3], the author established several results about the existence and the stability in the α-norm for neutral partial functional differential equations.

Let us recall some results that will be used throughout this work. Assume that,
(H1) $-A$ is the infinitestimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on a Banach space X and $0 \in \rho(A)$ where $\rho(A)$ is the resolvent set of A.

Then, there exist constants $M \geq 1$ and $\omega \in \mathbf{R}$ such that $||T(t)|| \leq Me^{\omega t}$ for $t \geq 0$. Without loss of generality, we assume that $\omega > 0$. If the assumption $0 \in \rho(A)$ is not satisfied, one can substitute the operator A for the operator $(A - \sigma I)$ with σ large enough so that $0 \in \rho(A - \sigma I)$ and so we can always assume that $0 \in \rho(A)$.

For the fractional power $(A^\alpha, D(A^\alpha))$, for $0 < \alpha < 1$, and its inverse $A^{-\alpha}$, one has the following known result.

THEOREM 2.2. ([19]) *Let $0 < \alpha < 1$ and assume that (H1) holds. Then*

(i) *$D(A^\alpha)$ is a Banach space with the norm $||x||_\alpha = ||A^\alpha x||$ for $x \in D(A^\alpha)$,*
(ii) *$T(t) : X \to D(A^\alpha)$for $t > 0$,*
(iii) *$A^\alpha T(t)x = T(t)A^\alpha x$ for $x \in A^\alpha$ and $t \geq 0$,*
(iv) *for every $t > 0, A^\alpha T(t)$ is bounded on X and there exists $M_\alpha > 0$ such that*
$$||A^\alpha T(t)|| \leq M_\alpha \frac{e^{\omega t}}{t^\alpha} \text{ for } t > 0,$$
(v) *$A^{-\alpha}$ is a bounded linear operator on X with $D(A^\alpha) = Im(A^{-\alpha})$,*
(vi) *if $0 < \alpha < \beta < 1$, then $D(A^\beta) \to D(A^\alpha)$,*
(vii) *there exists $N_\alpha > 0$ such that*
$$||(T(t) - I)A^{-\alpha}|| \leq N_\alpha t^\alpha \text{ for } t > 0.$$

In the sequel, we denote by X_α the Banach space $D(A^\alpha, ||\cdot||_\alpha)$. Recall that $A^{-\alpha}$ is given by the following formulas
$$A^{-\alpha} = \frac{sin(\pi\alpha)}{\alpha} \int_0^\infty t^{-\alpha}(tI + A)^{-1}dt$$
or
$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1}T(t)dt.$$
Both integrals converge in the uniform operator topology. Consequently, if $T(t)$ is compact for every $t > 0$, then A^α is compact for every $0 < \alpha < 1$.

Moreover, if $0 < \alpha < \beta < 1$, *then* $A^{-\beta} : X \to X_\alpha$ *is also compact.*

From now on, we use an axiomatic definition of the phase space B which was first introduced by Hale and Kato in [16]. We assume that B is the normed space of functions mapping $(-\infty, 0]$ into X and satisfying the following fundamental axioms:

(A) there exist a positive constant N, a locally bounded function $M(\cdot)$ on $[0,\infty)$ and a continuous function $K(\cdot)$ on $[0,\infty)$, such that if $x : (-\infty, a] \to X$ is continuous on $[\sigma, a]$ with $x_\sigma \in B$, for some $\sigma < a$, then for all $t \in [\sigma, a]$,
  (i) $x_t \in B$
  (ii) $t \to x_t$ is continuous with respect to $||\cdot||$ on $[\sigma, a]$,
  (iii) $N|x(t)| \leq ||x_t||_B \leq K(t-\sigma)\sup_{\sigma\leq s\leq t}|x(s)| + M(t-\sigma)||x_\sigma||_B$.

(B) B is a Banach space.

Lemma 2.3 ([17]). *Let* C_{00} *be the space of continuous funtions mapping* $(-\infty, 0]$ *into* X *with compact supports and* C_{00}^a *be the subspace of functions with supports included in* $[-a, 0]$ *endowed with the uniform topology. Then* $C_{00}^a \to B$.

Let $B_\alpha = \{\phi \in B : \phi(\theta) \in D(A^\alpha) \text{ for } \theta \leq 0 \text{ and } A^\alpha\phi \in B\}$ and provided B_α with the following norm

$$||\phi||_{B_\alpha} = ||A^\alpha\phi||_B \text{ for } \phi \in B_\alpha.$$

(H_2) $A^{-\alpha}\phi \in B$ for $\phi \in B$, where the function $A^{-\alpha}\phi$ is defined by

$$(A^{-\alpha}\phi)(\theta) = A^{-\alpha}(\phi(\theta)) \text{ for } \theta \leq 0$$

Lemma 2.4 ([17]). *Assume that* (H_1) *and* (H_2) *hold. Then* B_α *is a Banach space.*

3. Main theorem

Theorem 3.1. *Assume that the functions* f *and* g *satisfy assumptions (A),* $(H_1), (H_2), (A_1) - (A_5)$. *Then the nonlocal Cauchy problem (1.3)-(1.4) has a unique mild solution.*

Proof. Let $a > 0$ and $C([0, a] : X_\alpha)$ be a set of continuous functions and X_α provided with the uniform topology.
For $\phi \in B_\alpha$, we define the set

$$\Lambda = \{u \in C([0, a] : X_\alpha) : u(0) = \phi(0)\}.$$

Let $u \in \Lambda$ and $\tilde{u}$ an extension of u on $(-\infty, a]$ by

$$\tilde{u} = \begin{cases} u(t), & t \in [0,a] \\ \phi(t), & t \leq 0. \end{cases}$$

Let P be the operator defined on Λ by

$$\begin{aligned} P(u)(t) &= T(t)\phi(0) - T(t)[g(u_{t_1},...,u_{t_p})](0) \\ &\quad + \int_0^t T(t-s)f(s,u_s,\int_0^s k(s,\theta,u_\theta)d\theta)ds, t \in [0,a]. \end{aligned}$$

We claim that $P(\Lambda) \subset \Lambda$. In fact, let $u \in \Lambda, t_0 \in [0,a]$ and $t_0 < t < a$. Then

$$\begin{aligned} &A^\alpha(P(u)(t) - P(u)(t_0)) \\ &= T(t)A^\alpha\phi(0) - T(t_0)A^\alpha\phi(0) \\ &\quad + \int_0^{t_0} A^\alpha(T(t-s) - T(t_0-s))f(s,u_s,\int_0^s k(s,\theta,u_\theta)d\theta)ds \\ &\quad + \int_{t_0}^t A^\alpha T(t-s)f(s,u_s,\int_0^s k(s,\theta,u_\theta)d\theta)ds, t \in [0,a]. \end{aligned}$$

Since

$$T(t)A^\alpha\phi(0) - T(t_0)A^\alpha\phi(0) \to 0 \text{ as } t \to t_0$$

and

$$\begin{aligned} &\int_0^{t_0} A^\alpha(T(t-s) - T(t_0-s))f(s,u_s,\int_0^s k(s,\theta,u_\theta)d\theta)ds \\ &= (T(t-t_0) - I)\int_0^{t_0} A^\alpha T(t_0-s)f(s,u_s,\int_0^s k(s,\theta,u_\theta)d\theta)ds, \end{aligned}$$

it follows that

$$\int_0^{t_0} A^\alpha(T(t-s) - T(t_0-s))f(s,u_s,\int_0^s k(s,\theta,u_\theta)d\theta)ds \to 0 \text{ as } t \to t_0.$$

Moreover,

$$\int_{t_0}^t ||A^\alpha T(t-s)f(s,u_s,\int_0^s k(s,\theta,u_\theta)d\theta)||ds \to 0 \text{ as } t \to t_0.$$

Consequently,

$$A^\alpha(P(u)(t) - P(u)(t_0)) \to 0 \text{ as } t \to t_0 \text{ and } t > t_0.$$

Arguing as above, one can show that if $t_0 > 0$, then,

$$A^\alpha(P(u)(t) - P(u)(t_0)) \to 0 \text{ as } t \to t_0 \text{ and } t < t_0.$$

This implies that $P(u) \in \Lambda$ for all $u \in \Lambda$. In order to show that P has a unique fixed point in Λ, we use the strict contraction principle.

In fact, let $w, u \in \Lambda$ and $t \in [0, a]$. Then,

$$
\begin{aligned}
&(P(w)(t) - P(u)(t)) \\
&= -T(t)[g(w_{t_1}, ..., w_{t_p})(0) - g(u_{t_1}, ..., u_{t_p})(0)] \\
&\quad + \int_0^t T(t-s)[f(s, w_s, \int_0^s k(s, \theta, w_\theta)d\theta) - f(s, u_s, \int_0^s k(s, \theta, u_\theta)d\theta)]ds
\end{aligned}
$$

Taking the α-norm, we obtain

$$
\begin{aligned}
&||(P(w)(t) - P(u)(t))||_\alpha \\
&\leq ||T(t)||_\alpha ||[g(w_{t_1}, ..., w_{t_p})(0) - g(u_{t_1}, ..., u_{t_p})(0)]||_\alpha \\
&\quad + \int_0^t ||T(t-s)||_\alpha ||[f(s, w_s, \int_0^s k(s, \theta, w_\theta)d\theta) - f(s, u_s, \int_0^s k(s, \theta, u_\theta)d\theta)]||_\alpha ds \\
&\leq ||T(t)||_\alpha ||[g(w_{t_1}, ..., w_{t_p})(0) - g(u_{t_1}, ..., u_{t_p})(0)]||_\alpha \\
&\quad + \int_0^t ||T(t-s)||_\alpha [L||w-u||_\alpha + ||\int_0^s k(s, \theta, w_\theta)d\theta) - \int_0^s k(s, \theta, u_\theta)d\theta||_\alpha]ds \\
&\leq M_\alpha \frac{e^{\omega t}}{t^\alpha} G||w-u||_\alpha + M_\alpha L \int_0^t \frac{e^{\omega(t-s)}}{(t-s)^\alpha}[||w-u||_\alpha + aK||w-u||_\alpha]ds \\
&\leq M_\alpha L_a[1 + (1+aK)\int_0^a \frac{e^{\omega s}}{s^\alpha} ds]||w-u||_\alpha,
\end{aligned}
$$

where $L_a = \max\{\frac{e^{\omega t}}{t^\alpha} G, L\}$ and $||w-u||_\alpha$ denotes the supremum norm in $C([0, a] : X_\alpha)$. If we choose a such that

$$
M_\alpha L_a[1 + (1+aK)\int_0^a \frac{e^{\omega s}}{s^\alpha} ds]||w-u||_\alpha < 1,
$$

then P is a strict contractioin on Λ and it has a unique fixed point x which is the unique mild solution of equation on $(-\infty, a]$. □

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